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A NEW METHOD OF SOLVING NUMERICAL EQUATIONS.

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AMONG the methods for solving numerical equations, there are such as teach, to find from one or more initial approximate values a more approximate one of the root required. Such methods are, e. g., the method of Newton and the rule of false position. These methods become in certain cases impracticable because of slow convergence. The Newtonion method, e. g., converges very slowly toward a multiple root. For in the case of a simple root the convergence is of quadratic order, while in the case of a multiple root the acceleration is only linear. That is to say: in case of a simple root every correction is proportional to the square of the preceding correction; but in case of a multiple root every correction is proportional to the first power of the preceding correction.

It is true, the multiple roots can always be separated from the given equation, but the troublesome case of nearly equal roots remains. Besides this, the labor of separating multiple roots is not always small, and it is therefore desirable to avoid it, if possible.

The object of this essay is, to communicate a method of calculation which is independent of the relations of roots to one another, to wit, a method which converges with the same rapidity whether the roots be simple or multiple: The acceleration towards the roots is in every case of quadratic order. This algorithm is the following:

Theorem. Let $f(x)=0$ be a numerical equation of the n^{th} degree, x_0 an arbitrary approximate value of one of its roots, $f'(x)$ and $f''(x)$ the first and second differential quotients, then the next corrected value x_1 is found to be

$$x_1 = x_0 - \frac{f(x_0)f'(x_0)}{[f'(x_0)]^2 - f(x_0)f''(x_0)}, \tag{1}$$

or briefly

$$x_1 = x_0 - \frac{ff'}{(f')^2 - ff''} = F(x_0). \tag{2}$$

The next corrected value x_2 would be the same function of x_1 as x_1 is of x, etc.

Now let x be one of the roots of the given equation $f(x)=0$. In this case we would derive from equation (2) $x_1 = x$; for the correction would vanish, as $f(x)$ is a factor of it. Further, let us suppose x_0 to be an approximate value of this root, so that

$$x_0 = x + e,$$

then the corrected value x_1 would no more be equal to x_0, but we would have, according to (2):

$$x_1 = F(x + e) = F(x) + e.F'(x) + \tfrac{1}{2}e^2F''(x) + \tfrac{1}{6}e^3F'''(x) + \ldots\ldots$$

and x_1 would differ from the root by some quantity e_1, therefore

$$x_1 = x + e_1 = F(x) + e.F'(x) + \tfrac{1}{2}e^2F''(x) + \ldots\ldots$$

Now $x = F(x)$, consequently we get an expression for e_1,

$$e_1 = e.F'(x) + \tfrac{1}{2}e^2F''(x) + \tfrac{1}{6}e^3F'''(x) + \ldots.$$

The function $F'(x)$, the first differential quotient of $F(x)$ possesses the property (and this is essential for the establishment of our method) that, *the first differential quotient, $F'(x)$, vanishes for $f(x) = 0$, even in the case when the root x is a multiple root of any order.* Therefore the error e_1 of the corrected value of the root x_1 is

$$e_1 = \tfrac{1}{2}e^2F''(x) + \tfrac{1}{6}e^3F'''(x) + \ldots;$$

to wit, *the new error e_1 is proportional to the square of the preceding error,* whether the root in question is *simple* or *multiple;* that is, the acceleration of our method is always of second order, whereas the method of Newton,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)},$$

in case of a multiple root furnishes a convergence of linear order.

If it should happen that $F'''(x)$ vanishes for $f(x) = 0$, the acceleration would be of the third order.

This algorithm is only one particular case of an infinite number of methods of every possible rate of convergence. But the amount of work connected with those algorithms of greater convergence is not fully balanced by the greater rapidity of convergence. For example, an algorithm of cubic acceleration, for simple or multiple roots, is the following:

$$x_1 = x_0 - \frac{f(f'^2 - ff'')}{(f')^3 - \frac{1}{2}ff'f'' + \frac{1}{2}f^2f''},$$

where f, f', f'' are functions of x_0, f' the first, and f'' the second differential quotients of the given equation $f(x) = 0$.

As it would require considerable space to prove the above statements, I suppress the demonstration here, but I may be permitted to add one example:

Ex. Given the equation $(x - 2)^3 (x - 4) = 0$ with the multiple root 2.

Taking for initial value $x_0 = 3$, we find successively

$$x_1 = 2.5, \quad x_2 = 2.07, \quad x_3 = 2.0009, \quad x_4 = 2.0000001, \text{ etc.}$$

Observation. Always that root is found the modulus of which is the nearest to the modulus of the arbitrary initial trial value. If in the foregoing example we would commence with $x_0 = 5$, we would approach the root 4.